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ON SIMPLE PERIODIC LINEAR GROUPS— DENSE SUBGROUPS, PERMUTATION REPRESENTATIONS, AND INDUCED MODULES

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To John Thompson
in recognition of his many outstanding contributions to group theory

ABSTRACT

We determine the Zariski-dense subgroups of Chevalley groups and their twisted analogues over infinite algebraic extensions of finite fields. It turns out that these are essentially forms of the same group (possibly becoming twisted) over smaller infinite fields. It follows from our classification that if $\bar{G}$ is a simple algebraic group over the algebraic closure of a finite field, then a dense subgroup of $\bar{G}$ can never be maximal, and so the maximal subgroups of $\bar{G}$ are necessarily closed. It follows that Seitz's determination of the closed maximal subgroups of $\bar{G}$ actually gives all the maximal subgroups.

It also enables us to prove that if G is a simple Chevalley group or twisted type over an infinite algebraic extension of a finite field, then in every non-trivial permutation representation of G, every finite subgroup has a regular orbit. It follows that every non-trivial permutation module for G over a field k is kG-faithful. This is relevant to a programme of studying ideals in group rings of simple locally finite groups.

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1. Introduction

The first part of this paper consists of the determination of the Zariski-dense subgroups of Chevalley groups and their twisted versions over infinite locally finite fields. It turns out that these are essentially forms of the same group (possibly becoming twisted) over smaller infinite fields. This result depends on the classification of infinite simple periodic linear groups, and through that, on the classification of finite simple groups. It will follow from our work that if $\overline{K}$ is the algebraic closure of a finite field and $\overline{G}$ is a simple algebraic group over $\overline{K}$, then a dense subgroup of $\overline{G}$ can never be maximal, and so the maximal subgroups of $\overline{G}$ are necessarily closed. Thus, Seitz's determination of the closed maximal subgroups of $\overline{G}$ [15, 16] actually gives all the maximal subgroups.

Let G be a simple Chevalley group of normal or twisted type over an infinite subfield of $\overline{K}$. We were led to study dense subgroups of G by a desire to show that G has no proper "enormous" subgroups, as conjectured by the second author in work on locally finite groups whose complex group rings have simple augmentation ideals [22]. These subgroups, which we call **confined subgroups** in the sequel, are a generalization of normal subgroups, and so their absence is a strong form of simplicity. We show that G has no such subgroups. This tells us that in every non-trivial permutation representation of G, every finite subgroup has a regular orbit, and hence that if R is any commutative ring, then every non-zero R-free RG-module induced from a proper subgroup of G is RG-faithful.

To state our results precisely, we need to introduce more notation. Let $\overline{K}$ be the algebraic closure of the field of p elements, and let $\overline{G}$ be a simple algebraic group over $\overline{K}$ (meaning that $\overline{G}$ is simple as algebraic group). We shall always think of $\overline{G}$ in rather concrete terms as a matrix group constructed as in Steinberg's notes [17] from a finite dimensional simple Lie algebra over the complex field and a finite-dimensional representation of it. Thus, $\overline{G}$ comes equipped with a group Φ of field automorphisms and a set Γ consisting of 1, together, if the conditions are right, with certain canonical graph automorphisms constructed as in [17] or [6] Let $A = \Phi\Gamma$. By combining, in various ways, the elements of Γ with powers of the field automorphism corresponding to $\lambda \mapsto \lambda^p$, we obtain Frobenius maps in A. If F is any such Frobenius map, let $\overline{G}^F$ be its fixed point group. Let $\mathbf{N}$ be the set of all unbounded sequences

$$\mathbf{n} = (n_1, n_2, \dots)$$

of natural numbers such that $n_1 \mid n_2 \mid \ldots$, and for $\mathbf{n} \in \mathbf{N}$, let

$$\overline{G}(F,\mathbf{n}) = \bigcup_{i=1}^{\infty} \overline{G}^{F^{n_i}}.$$

Then the groups $\overline{G}(F,\mathbf{n})'$ give us the various quasisimple Chevalley groups and their twisted versions, of the same "untwisted type" as $\overline{G}$, over the various infinite subfields of $\overline{K}$. A perfect subgroup of $\overline{G}$ of the form $\overline{G}(F,\mathbf{n})'$, for some Frobenius map $F \in A$ and $\mathbf{n} \in \mathbf{N}$, will be said to be in **standard position**. In particular, $\overline{G}$ itself is in standard position. See [12] and the next section for further discussion of this. If $\overline{G}$ is of adjoint type, then $\overline{G}(F,\mathbf{n})$ is simple and it is known [1, 5, 10, 19] that every infinite periodic simple linear group is isomorphic to such a group.

As a $\overline{K}$-linear group, $\overline{G}(F,\mathbf{n})$ has a Zariski topology, inherited from the Zariski topology of the appropriate $GL_n(\overline{K})$. There is a certain subfield K of $\overline{K}$ naturally associated with $\overline{G}(F,\mathbf{n})$, and $\overline{G}(F,\mathbf{n})$ is also K-linear, so it also has a K-Zariski topology; however these topologies are identical [21, p. 73]. For definiteness, we think of all Zariski topologies as associated with $\overline{K}$. It is presumably common knowledge that the groups $\overline{G}(F,\mathbf{n})'$ are all dense in $\overline{G}$ (see [11] for a proof). Now we can state a converse.

THEOREM A: *Let G be a perfect subgroup of $\overline{G}$ in standard position, and let H be a subgroup of G. Then H is Zariski dense in G if and only if there exist elements $\mathbf{m},\mathbf{n} \in \mathbf{N}$, a Frobenius map $F \in A$, and an automorphism ϕ of $\overline{G}(F,\mathbf{n})$ such that $G = \overline{G}(F,\mathbf{n})'$, $\mathbf{m} \mid \mathbf{n}$ and*

$$\overline{G}(F,\mathbf{m})' \leq \phi(H) \leq N_{\overline{G}}(\overline{G}(F,\mathbf{m})').$$

If $\overline{G}$ has adjoint type, then $N_{\overline{G}}(\overline{G}(F,\mathbf{m})') = \overline{G}(F,\mathbf{m})$, and in any case, $N_{\overline{G}}(\overline{G}(F,\mathbf{m})')/\overline{G}(F,\mathbf{m})$ is isomorphic to a subgroup of the fundamental group of $\overline{G}$.

By the notation $\mathbf{m} \mid \mathbf{n}$, we understand that given i, there exists j such that $m_i \mid n_j$. Except in type D_4, we may take ϕ to be the product of an inner and a diagonal automorphism, and thus to be algebraic, since the field and graph automorphisms of $\overline{G}$ normalize the groups $\overline{G}(F,\mathbf{m})$. In type D_4 we may need a graph automorphism, but these are also algebraic. The "if" part of the theorem follows from this and the remarks preceding its statement.

Thus, for instance, the dense subgroups of $PSL_n(K)$, where K is infinite and locally finite, are conjugates in $PGL_n(K)$ of subgroups of $PSL_n(K)$ situated as follows: (i) between $PSL_n(k)$ and $PGL_n(k)$, where k is an infinite subfield of K, and (ii) (if $n \geq 3$) between $PSU_n(k)$ and $PGU_n(k)$, where k is an infinite subfield of K admitting an automorphism of order 2. Note that in order to mention the dense unitary subgroups of $PSL_n(K)$ in the above notation, we must think of $PSL_n(K)$ as specified by powers of a twisted Frobenius map; this explains the rather stilted form of the statement of the theorem. We note that Steinberg [17, Lemma 77] describes homomorphisms from one simple Chevalley group to another with dense image, in the case when the field over which the target group is defined is algebraically closed but not necessarily locally finite.

COROLLARY A1: *Suppose that $\overline{G}$ has adjoint type. Let $G = \overline{G}(F, \mathbf{n})'$, let H be a dense subgroup of G, and let $\mathbf{m}$ and ϕ be as in Theorem A. Let K be the field over which G is defined and L be that over which $\overline{G}(F, \mathbf{m})'$ is defined. Let $H_1 = N_G(H')$. Then H_1/H is finite abelian, and one of the following happens.*
(i) $[K : L] < \infty$, and there exists an automorphism τ of finite order of G such that $C_G(\tau) = H_1$.
(ii) $[K : L] = \infty$, and there exists a chain of subgroups

$$H_1 < H_2 < \cdots$$

such that $\bigcup_{i=1}^{\infty} H_i = G$ and $H_1 = C_{H_i}(\tau_i)$ for some automorphism τ_i of finite order of H_i.

It is more or less clear what we mean by the field over which $\overline{G}(F, \mathbf{n})'$ is defined, but we shall spell it out in the next section.

When $K = \overline{K}$, and H is proper, case (i) cannot arise, as $\overline{K}$ is not a finite extension of any proper subfield, and so as previously stated, we obtain the following.

COROLLARY A2: *Every maximal subgroup of $\overline{G}$ is closed.*

The following curious result will be important to us in the sequel.

COROLLARY A3: *Let G, H and H_1 be as in Corollary A1, with $H \neq G$, and let X be any finite subset of G. Then there exists a subgroup D of G such that $< H_1, X > \leq D$, and an embedding of D in some $\overline{K}$-linear group under which H_1 corresponds to a closed subgroup.*

As already stated, our interest in these matters arose from ideas of the second author in investigating simple locally finite groups whose group rings over some field K have precisely three ideals. We call such groups **K-augmentation simple**, omitting explicit reference to the field if the context allows it. The question of describing such group rings goes back to Kaplansky [13], and some examples were given in [2]. The following striking result is now known.

THEOREM: *If G is an infinite simple periodic linear group, then G is $\mathbb{C}$-augmentation simple.*

The second author formulated a conjecture on values of irreducible characters of finite groups of Lie type, from which this assertion would follow [22]. This has now been established by Gluck [8], [9]. For more information on these matters, see [22, 24].

Now annihilators of permutation modules form a natural family of ideals of a group ring, and so in the present context, it is reasonable to look for conditions on a subgroup H of a group G, under which the permutation module on the cosets of H is faithful. One such condition arises from the following definition.

Definition: A subgroup H of a group G is called **confined**, if there exists a finite subset F of $G \setminus 1$ such that $H^g \cap F \neq \emptyset$ for all $g \in G$.

Confined subgroups are called **enormous** in [14]. Clearly H is confined if it contains a non-trivial normal subgroup of G; also every non-trivial subgroup of a finite group is confined. The trivial subgroup is not confined. Infinite locally finite simple groups can have proper confined subgroups, and indeed infinite alternating groups have them [14], but this seems to be the exception. The following easy result, essentially in [14], explains their relevance.

If R is any commutative ring , H is a subgroup of a group G, and L is a non-zero R-free RH-module such that the induced module L^G has non-trivial annihilator, then H is confined in G.

For a slightly stronger version, see Lemma 6.3.

It seems reasonable to conjecture that a simple locally finite group is augmentation simple, say over the complex field, if and only if no proper subgroup is confined. For linear groups, this is confirmed by our second main result, taking into account the theorem above.

THEOREM B: *If G is an infinite periodic simple linear group, then no proper subgroup of G is confined.*

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2. Preliminaries on Chevalley groups

This section is mostly devoted to discussing further the notation we are using for these groups. We take for granted the standard facts, as found for example in [17] or [6], and leave the verification of a number of routine statements below to the reader. We have introduced the subsets Φ, Γ and A of $\mathrm{Aut}(\overline{G})$ in the Introduction, and we write $O^{p'}(X)$ for the group generated by the p-elements of a group X. If the Frobenius map $F \in A$ corresponds to a symmetry of order r of the Dynkin diagram of $\overline{G}$, then F^r is a field automorphism of $\overline{G}$, and we let $K(F)$ denote the fixed field of this field automorphism. By the first part of the next result, this depends only on the group $O^{p'}(\overline{G}^F)$. If $\overline{G}^F$ is twisted, then $\overline{G}^{F^r}$ is its "untwisted version," and $K(F) = K(F^r)$. (We use words like "automorphism" to refer to the abstract group structure, otherwise using "polynomial automorphism" or "algebraic automorphism".)

The field assumptions in the second part of the next result are certainly stronger than necessary. Since we are really interested in infinite groups, this is not an important issue for us, and so we have not pursued it. Further, conjugacy in that result will be equality except in type D_4.

LEMMA 2.1: *Let E and F be Frobenius maps in A.*
(i) $O^{p'}(\overline{G}^E) \geq O^{p'}(\overline{G}^F)$ *if and only if E is a power of F.*
(ii) *If $|K(F)| \geq 64$ and $O^{p'}(\overline{G}^F)$ is isomorphic to a subgroup of $O^{p'}(\overline{G}^E)$, then E conjugate in A to a power of F.*

Proof: (i) This can easily be verified by using the form of the root subgroups in the various cases.
(ii) Let $G = \overline{G}^E$, and let G_1 be a subgroup of G isomorphic to $\overline{G}^F$. Let U be a Sylow p-subgroup of G, $B = N_G(U)$, let H be a Hall p'-subgroup of B, and let $N = N_G(H)$. Let U_1, B_1, H_1 and N_1 be similarly defined with respect to G_1, and suppose without loss of generality that $U_1 \leq U$. Now U and U_1 have the same nilpotency class, while two distinct G-conjugates of U intersect in a group of smaller class [10, Lemmas 2.1 and 4.11] see also [11, Proof of Lemma 3.4], and

so

$$B_1 \leq B.$$

By Hall's Theorem, we may assume that

$$H_1 \leq H.$$

Now $K(E)$ is generated by elements whose multiplicative orders are orders of elements of H, and similarly for $K(F)$ [10, p.52], whence

$$K(F) \leq K(E). \tag{1}$$

In particular, $|K(E)| \geq 64$. Arguing as in [10, p.53], we find that $N \cap G_1 = N_1$, and so N_1/H_1 is isomorphic to a subgroup of N/H. Now the fields are sufficiently large to ensure that N/H and N_1/H_1 are isomorphic to the Weyl groups of G and $\overline{G}^F$ respectively, so if W is the Weyl group of $\overline{G}$, then W^F is isomorphic to a subgroup of W^E. Thus, if F acts non-trivially on the Dynkin diagram of $\overline{G}$, then so does E, and by considering the twisted Weyl groups of type D, we see that in that case, E and F induce symmetries of the same order on the Dynkin diagram. Replacing E by a conjugate in A, we may assume in that case that E and F induce the same symmetry of the Dynkin diagram.

Now we must consider various cases. If E and F both act trivially on the Dynkin diagram, then $E = F_0^m$ and $F = F_0^n$, where F_0 is the field automorphism corresponding to $\lambda \mapsto \lambda^p$. Then (1) gives $n \mid m$, and so E is a power of F. As is noted above, if one of E and F acts non-trivially on the Dynkin diagram, then F does. In that case, let the symmetry of the Dynkin diagram induced by F have order r. Then $F^r = F_0^{tr}$ for some integer $t \geq 1$, where F_0 is as above, and $|K(F)| = p^{tr}$. It follows from (1) that $|K(E)| = p^{mtr}$ for some integer $m \geq 1$. If E acts trivially on the Dynkin diagram, this means that $E = F_0^{mtr}$. Then $F^{mr} = F_0^{mtr} = E$, as required.

There remains the case when both E and F act non-trivially on the Dynkin diagram. In the Suzuki or Ree group cases, both E and F are powers of a basic graph automorphism, and (1) is sufficient to yield the result, as in the untwisted case. Thus, we are reduced to the cases when $\overline{G}$ has type A, D, E_6. Then, by [10, Theorem C$'$], we may take G_1 to be in standard position, that is, $G_1 = \overline{G}^D$ for some Frobenius map $D \in A$. By (1), $K(D) = K(F)$. It follows that if $|K(F)| = q$, then $|K(E)| = q^t$, where $(t, r) = 1$. Since E and F induce the same symmetry of the Dynkin diagram, we may write $E = F_0^a\sigma$ and $F = F_0^b\sigma$, where σ is an element of finite order in A; of course, σ commutes with F_0. Then $K(E) = K(E^r)$ has order p^{ra}, and $K(F)$ has order p^{rb}, whence from the above, $a = tb$. Then $F^t = F_0^{tb}\sigma^t = F_0^a\sigma^t$. Since $(t,r) = 1$, we have $\sigma^t = \sigma$ except perhaps in type D_4 when $r = 3$, and in that case, σ^t is conjugate to σ under an element of A that commutes with F_0. Thus, F^a is conjugate in A to E. ∎

We defined the notation $\mathbf{m} \mid \mathbf{n}$ ($\mathbf{m}, \mathbf{n} \in \mathbf{N}$) in the Introduction. We define the equivalence relation $\sim$ on $\mathbf{N}$ by requiring that $\mathbf{m} \sim \mathbf{n}$ if and only if $\mathbf{m} \mid \mathbf{n}$ and $\mathbf{n} \mid \mathbf{m}$. For an integer $l \geq 1$ and $\mathbf{m} \in \mathbf{N}$, we define $l\mathbf{m} = (lm_1, lm_2, \ldots)$. We put

$$\overline{G}(F,\mathbf{n}) = \bigcup_{i=1}^{\infty} \overline{G}^{F^{n_i}},$$

and we have of course

$$\overline{G}(F,\mathbf{n})' = \bigcup_{i=1}^{\infty} O^{p'}(\overline{G}^{F^{n_i}}).$$

It is clear that we may extract a subsequence of $\mathbf{n}$ such that the maps F^{n_i} corresponding to that subsequence all determine the same symmetry of the Dynkin diagram. We shall always assume that has been done. We may also assume that $n_1 = 1$ at will.

LEMMA 2.2: *Let E and F be Frobenius maps in A, and let $\mathbf{m}, \mathbf{n} \in \mathbf{N}$. Then*
(i) *$\overline{G}(F,\mathbf{n})'$ is isomorphic to a subgroup of $\overline{G}(E,\mathbf{m})'$ if and only if, for each i, there exists j such that E^{m_j} is a power of F^{n_i}.*
(ii) *$\overline{G}(F,\mathbf{n})' = \overline{G}(F,\mathbf{m})'$ if and only if $\mathbf{n} \sim \mathbf{m}$. In that case, $\overline{G}(F,\mathbf{n}) = \overline{G}(F,\mathbf{m})$.*

Proof: (i) Assume that $\overline{G}(F,\mathbf{n})'$ is isomorphic to a subgroup of $\overline{G}(E,\mathbf{m})'$. We may without loss of generality consider only those i such that $|K(F^{n_i})| \geq 64$. For each such i, there exists j such that $\overline{G}^{F^{n_i}}$ is isomorphic to a subgroup of $\overline{G}^{E^{m_j}}$. By Lemma 2.1, E^{m_j} is a power of F^{n_i}.

The converse is clear, and (ii) follows. ∎

It follows that if $\overline{G}(F,\mathbf{n})' = \overline{G}(F,\mathbf{m})'$, then $\bigcup_{i=1}^{\infty} K(F^{n_i}) = \bigcup_{j=1}^{\infty} K(F^{m_j})$. We denote this field by $K(\overline{G}(F,\mathbf{n})')$, or often simply by $K(\mathbf{n})$.

If K is any subfield of $\overline{K}$ and n is the size of the matrices constituting $\overline{G}$, we write $\overline{G}_K = \overline{G} \cap GL_n(K)$ for the group of K-rational points of $\overline{G}$, and $\overline{G}(K) = O^{p'}(\overline{G}_K) = (\overline{G}_K)'$ for the untwisted Chevalley group of the same type as $\overline{G}$ over

K. Let $\mathbf{n} \in \mathrm{N}$ and $K = K(\overline{G}(F,\mathbf{n})')$. If all F^{n_i} are field automorphisms,then $\overline{G}_K = \overline{G}(F,\mathbf{n})$. If all F^{n_i} correspond to a symmetry of order 2 of the Dynkin diagram, then $\overline{G}_K = \overline{G}(F,2\mathbf{n})$. The map F^{n_i} induces an automorphism σ_i of order 2 of $\overline{G}^{F^{2n_i}}$. These automorphisms are consistent under restriction, and determine an automorphism of order 2 of $\overline{G}(F,2\mathbf{n})$, induced by an element of A, whose unipotent fixed points generate $\overline{G}(F,\mathbf{n})'$. The situation is similar if $\overline{G}$ has type D_4 and the symmetry of the Dynkin diagram corresponding to the F^{n_i} has order 3.

LEMMA 2.3: *There exists an automorphism $\sigma \in A$ such that $C_{\overline{G}}(\sigma) = \overline{G}(F,\mathbf{m})$ and $\sigma\Phi = F\Phi$.*

Proof: If F is untwisted, then $\overline{G}(F,\mathbf{m}) = \overline{G}_K$, where $K = K(\mathbf{m})$. There is an automorphism of $\overline{K}$ whose fixed field is K, and we take σ to be the corresponding field automorphism of $\overline{G}$. If $\overline{G}(F,\mathbf{m})$ has type ${}^2A_l, {}^2D_l$ or ${}^2E_6$, let $K = K(\mathbf{m})$ and let θ be an automorphism of $\overline{K}$ having as its fixed field the subfield K_0 of K such that $[K : K_0] = 2$. In order for these twisted groups to exist over K, there must be such a subfield. The fixed field of θ^2 is K, as we see by considering the restriction of θ to a finite subfield F of $\overline{K}$ such that $[K \cap F : K_0 \cap F] = 2$. Let $\Phi F = \Phi\delta$ with $\delta \in \Gamma$ and let σ be the graph-field automorphism $\theta\delta$. Then $C_{\overline{G}}(\sigma) \leq C_{\overline{G}}(\sigma^2) = \overline{G}_K$, and σ induces on $\overline{G}_K$ the automorphism whose fixed points form $\overline{G}(F,\mathbf{m})$. The same argument applies in type ${}^3D_4$.

Consider now the case when $\overline{G}(F,\mathbf{m})$ has type ${}^2B_2$ or ${}^2F_4$. (The same argument applies in type ${}^2G_2$.) If $K = K(\mathbf{m})$, then we have an automorphism θ of K satisfying $2\theta^2 = 1$. We wish to extend θ to an automorphism ψ of $\overline{K}$ such that the fixed field of $2\psi^2$ is K. For this, consider the set $\mathbf{F}$ of finite subfields of $\overline{K}$. If $F \in \mathbf{F}$, then $|K \cap F| = 2^{2a+1}$ for some a. The restriction of θ to $K \cap F$ is given by $\lambda \mapsto \lambda^{2^a}$ and if β is the automorphism of F given by the same formula , then $2\beta^2$ has fixed field $K \cap F$. Thus, if $S(F)$ is the set of all automorphisms β of F such that the fixed field of $2\beta^2$ (in F) is $K \cap F$, then $S(F)$ is a non-empty finite set. If $F_1, F_2 \in \mathbf{F}$ and $F_1 \geq F_2$, then restriction induces a map $S(F_1) \to S(F_2)$. Since any inverse system of non-empty finite sets has non-empty inverse limit, we can pick out elements $\theta_F \in S(F)\,(F \in \mathbf{F})$ that are consistent under restriction. These determine an automorphism ψ of $\overline{K}$ such that $2\psi^2$ has fixed field K. The automorphism $\sigma = \psi\gamma \in A$, where γ is the non-trivial element of Γ, has $\sigma^2 = 2\psi^2$ (with a little abuse of notation), so we easily find that its centralizer in $\overline{G}$ is

$\overline{G}(F, \mathbf{m})$. ∎

LEMMA 2.4: *Let* $\mathbf{m}, \mathbf{n} \in \mathrm{N}$. *Then the following statements are equivalent.*
(i) $\overline{G}(F, \mathbf{m}) \leq \overline{G}(F, \mathbf{n})$.
(ii) $\overline{G}(F, \mathbf{m})' \leq \overline{G}(F, \mathbf{n})'$.
(iii) $\overline{G}(F, \mathbf{m})'$ *is isomorphic to a subgroup of* $\overline{G}(F, \mathbf{n})'$.
(iv) $K(\mathbf{m}) \leq K(\mathbf{n})$.
(v) $\mathbf{m} \mid \mathbf{n}$.

Proof: We have trivially (v) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (v) by Lemma 2.2. We show that (iv) $\Leftrightarrow$ (v). We may assume that $m_1 = 1$. Let $|K(\overline{G}^F)| = q$, and let f be the order of the symmetry of the Dynkin diagram corresponding to F, and hence to all F^{m_i}. Then $|K(\overline{G}^{F^{m_j}})| = q^{m_j}$, and $|K(\overline{G}^{F^{n_i}})| = q^{n_i}$ if F^{n_1} corresponds to the same twist as F, or $q^{n_i/f}$ otherwise. In the first case it is clear that $K(\mathbf{m}) \leq K(\mathbf{n})$ if and only if $\mathbf{m} \mid \mathbf{n}$. In the second, note that $(m_j, f) = 1$ and $f \mid n_i$ for all i and j, so that $m_j \mid n_i$ if and only if $m_j f \mid n_i$. With this observation, the equivalence is clear in this case also. ∎

LEMMA 2.5: *Let* $\mathbf{m}, \mathbf{n} \in \mathrm{N}$, *and let* σ *be the element of* A *as constructed in Lemma 2.3, such that* $C_{\overline{G}}(\sigma) = \overline{G}(F, \mathbf{m})$. *Suppose that* $\mathbf{m} \mid \mathbf{n}$, *and let* $\sigma_{\mathbf{n}}$ *be the restriction of* σ *to* $\overline{G}(F, \mathbf{n})$. *Then the following are equivalent.*
(i) $[K(\mathbf{n}) : K(\mathbf{m})] < \infty$.
(ii) $\sigma_{\mathbf{n}}$ *has finite order.*
(iii) $\mathbf{n} \sim l\mathbf{m}$ *for some* $l \geq 1$.

Proof: We know by Lemma 2.4 that $\overline{G}(F, \mathbf{m}) \leq \overline{G}(F, \mathbf{n})$. First we show that (i) $\Leftrightarrow$ (ii). Let $\tau = \sigma^r$, where r is the order of $\Phi\sigma$ in A/Φ. Then from the construction, τ is an element of Φ whose fixed field is $K(\mathbf{m})$. Also, the restriction $\tau_{\mathbf{n}}$ of τ to $\overline{G}(F, \mathbf{n})$ has finite order if and only if $\sigma_{\mathbf{n}}$ does. Now by considering the action of $\tau_{\mathbf{n}}$ on suitable root subgroups, we see that $\tau_{\mathbf{n}}^k = 1$ as an automorphism of $\overline{G}(F, \mathbf{n})$ if and only if $\tau_{\mathbf{n}}^k = 1$ as an automorphism of $K(\mathbf{n})$. Thus, by Galois theory, this happens for some k if and only if $[K(\mathbf{n}) : K(\mathbf{m})] < \infty$.

To see that (ii) $\Leftrightarrow$ (iii), suppose first that the F^{m_i} and F^{n_j} both correspond to the trivial symmetry of the Dynkin diagram, and $m_1 = 1$. Then if $\mathbb{F}_q = K(\overline{G}^F)$, we have $K(\mathbf{m}) = \bigcup_{i=1}^{\infty} \mathbb{F}_{q^{m_i}}$ and $K(\mathbf{n}) = \bigcup_{j=1}^{\infty} \mathbb{F}_{q^{n_j}}$. The statement $\mathbf{n} \sim l\mathbf{m}$ amounts to saying that for each i, there exists j such that $\mathbb{F}_{q^{n_i}} \leq \mathbb{F}_{q^{lm_j}}$ and

$\mathbb{F}_{q^{lm_i}} \leq \mathbb{F}_{q^{n_j}}$. These statements are equivalent to saying that $K(\mathbf{n}) = \bigcup_{i=1}^{\infty} \mathbb{F}_{q^{lm_i}}$, and clearly $\bigcup_{i=1}^{\infty} \mathbb{F}_{q^{lm_i}}$ is a finite extension of $\bigcup_{i=1}^{\infty} \mathbb{F}_{q^{m_i}}$.

The other cases are similar. ∎

LEMMA 2.6: *Let $\mathbf{m}, \mathbf{n} \in \mathrm{N}$, and suppose that $\mathbf{m} \mid \mathbf{n}$. Let σ be the element of A given by Lemma 2.3, such that $C_G(\sigma) = \overline{G}(F, \mathbf{m})$. Suppose that $[K(\mathbf{n}) : K(\mathbf{m})] = \infty$. Then there exists $\mathbf{r} \in \mathrm{N}$ such that*

$$\overline{G}(F, \mathbf{m}) < \overline{G}(F, r_1\mathbf{m}) < \overline{G}(F, r_2\mathbf{m}) < \cdots$$

and

$$\bigcup_{j=1}^{\infty} \overline{G}(F, r_j\mathbf{m}) = \overline{G}(F, \mathbf{n}).$$

Proof: For fixed j, let n_{ij} be the least common multiple of m_i and n_j and x_{ij} their greatest common divisor. Then

$$x_{ij} \mid x_{i+1,j} \mid \ldots \mid n_j,$$

and so the sequence (x_{ij}) is eventually constant and equal say to h_j. Let $k_j = n_j/h_j$. Then, for all but finitely many values of $i, n_{ij} = m_i k_j$. As $n_j \mid n_{j+1}$, we have $n_{ij} \mid n_{i,j+1}$, and so $k_j \mid k_{j+1}$. Also, $k_j\mathbf{m} \mid \mathbf{n}$. Suppose if possible that there exists i such that $k_i\mathbf{m} \sim k_{i+1}\mathbf{m} \sim \ldots$. Then since $n_s \mid k_s m_t$, for some t, we find that if $s \geq i$, then $n_s \mid k_i m_j$ for some j,, whence $\mathbf{n} \sim k_i\mathbf{m}$. By Lemma 2.5, we find that $[K(\mathbf{n}) : K(\mathbf{m})] < \infty$, contrary to assumption.

Thus, there is a subsequence $(r_1, r_2, \ldots)$ of $(k_1, k_2, \ldots)$ such that $r_{i+1}\mathbf{m}$ does not divide $r_i\mathbf{m}$ for all i. By Lemmas 2.2 and 2.4, we obtain the desired conclusion. ∎

3. Some representation theory

We begin with some facts about tensor decompositions of modules. Some or all of this may be well known, but since we have been unable to find references, we give a detailed discussion. We go on to study the irreducible $\overline{K}$-representations of the groups $\overline{G}(F, \mathbf{n})$, where $\overline{G}$ is simply connected, and show, among other things, that they extend to $\overline{G}$. In many cases this follows from work of Borel and Tits [3], see also [17, Theorem 42], but their results do not seem to cover the Suzuki and Ree groups, for example, which are important for us. The proof is mostly a matter of putting together Steinberg's results on extending irreducible representations from $\overline{G}^F$ to $\overline{G}$.

Definition: Let K be a field, G a group and V a KG-module. Then V is **tensor-decomposable**, if there exist KG-modules U and W, both of dimension at least 2, such that $V \cong U \otimes_K W$. Otherwise, V is **tensor-indecomposable.**

Suppose that G is an algebraic group, V affords a rational representation, and we are told that V is tensor-decomposable as above. We wish to investigate whether U and W can be taken to afford rational representations. To discuss this, we need to set up some notation. Let K be infinite, and from here until the end of Lemma 3.3, let $G_n = GL_n(K)$. We index the rows and columns of elements of G_{mn} by pairs (i,j) $(1 \le i \le m, 1 \le j \le n)$, ordered in some way which will be unimportant to us. We have the map $\tau : G_m \times G_n \to G_{mn}$ which sends a pair (g,h) to the matrix whose $((i,j),(k,l))$ entry is $g_{ik}h_{jl}$. We write $\tau(g,h) = g \otimes h$, "the" Kronecker product of g and h, and $T = \operatorname{im}\tau = G_m \otimes G_n$. Then τ is a polynomial homomorphism whose kernel consists of the elements (λ, λ^{-1}) $(0 \ne \lambda \in K)$, and so $g \otimes h = g' \otimes h'$ if and only if $g' = \lambda g$ and $h' = \lambda^{-1}h$ for some non-zero $\lambda \in K$. Hence, if PG_m denotes $PGL_m(K)$, we have well defined projections $\phi_m : T \to PG_m$, and $\phi_n : T \to PG_n$, sending $g \otimes h$ to the images of g and h respectively.

Now PG_m is usually viewed as a matrix group by considering the conjugation action ρ of G_m on $A_m = M_m(K)$. We have $\rho(g)(a) = gag^{-1}$. The space A_m has the usual matrix units e_{ij} as basis, and with respect to this basis, linear transformations of A_m are represented by matrices $(u_{ij,kl})$. The matrix of $\rho(g)$ has (ij,kl)-entry

$$(\det g)^{-1} g_{ki}\hat{g}_{lj}, \tag{1}$$

where $\hat{g}_{jl}$ is the cofactor of g_{jl} in g. If $\pi_m : G_m \times G_n \to PG_m$ sends (g,h) to the matrix given by (1), then

$$\phi_m \tau = \pi_m, \tag{2}$$

and similarly with the second component. The maps π_m, π_n are rational homomorphisms.

LEMMA 3.1: *ϕ_m is a rational homomorphism.*

Proof: The matrix $\tau(g,h)$ has entries $g_{ik}h_{jl}$, where the suffices vary independently. Consequently, any expression $g_{i_1k_1}\cdots g_{i_rk_r}h_{j_1l_1}\cdots h_{j_rk_r}$, the product of two monomials of the same degree in the entries of g and h, is a monomial in the entries of $g \otimes h$. Hence the expression

$$(\det g)^{(n-1)} g_{ki} \hat{g}_{lj} (\det h)^m$$

is a polynomial in the entries of $g \otimes h$. For the same reason, so is $(\det g)^n (\det h)^m$. Dividing one by the other shows us that (1) is a rational function of the entries of $g \otimes h$, as required. ∎

LEMMA 3.2: *Continuing with the above notation, let G be a Chevalley group and $\alpha : G \to G_m$, $\beta : G \to G_n$ be representations. Suppose that $\alpha \otimes \beta : G \to T = G_m \otimes G_n \leq G_{mn}$ is polynomial. Then α and β are polynomial.*

We are grateful to S. Donkin for showing us another proof of this result. Conceivably it may be true more generally. Taking $G = K^*$ and α to be any automorphism of K, we have that $\alpha \otimes \alpha^{-1}$ is polynomial, while α need not be in general. It may be that for more general G, there exists a homomorphism $\lambda : G \to K^*$ such that $\lambda\alpha$ and $\lambda^{-1}\beta$ are rational.

Proof: We may assume that G is simply connected. Let $\theta = \alpha \otimes \beta$. By Lemma 3.1, $\phi_m\theta : G \to PG_m$ is a rational map (and so in fact polynomial). Let $\nu_m : G_m \to PG_m$ be the natural projection (which sends the matrix (g_{ij}) to the matrix with entries (1)). By (2), we have

$$\phi_m\theta(g) = \nu_m\alpha(g)$$

if $g \in G$, and the left hand side is a polynomial homomorphism. Now every polynomial projective representation of G lifts to an ordinary polynomial representation, since G is simply connected [17, p. 91], and so there exists a polynomial homomorphism $\alpha' : G \to G_m$ such that $\nu_m\alpha(g) = \nu_m\alpha'(g)$ if $g \in G$. Then α and α' differ by a one dimensional representation of G, and this must be trivial as G is perfect.

One can also argue that ν_m induces a polynomial isomorphism on each unipotent subgroup of G, from which it follows that the restriction of α to each root subgroup of G is polynomial, and hence, arguing for example as in [17, Theorem 30, p.158], that α itself is polynomial. The upshot is that α, and similarly β, is polynomial. ∎

LEMMA 3.3: *Continuing further with the same notation, let G be a simply connected Chevalley group and $\rho : G \to G_{mn}$ be a representation. Suppose that there exists $u \in G_{mn}$ such that $\rho(G)^u \leq T = G_m \otimes G_n$. Then there exist representations $\alpha : G \to G_m$ and $\beta : G \to G_n$ such that ρ is equivalent to $\alpha \otimes \beta$. If ρ is polynomial, so are α and β.*

Proof: Without loss of generality, $u = 1$. Since G is simply connected, there exist representations $\alpha : G \to G_m$ and $\beta : G \to G_n$ such that $\phi_m\rho = \nu_m\alpha$ and $\phi_n\rho = \nu_n\beta$. If $\rho(g)$ is written as $x \otimes y$ in any way, with $x \in G_m, y \in G_n$, then $\alpha(g) = \mu x$ and $\beta(g) = \mu' y$ for some scalars μ and μ', and so

$$\alpha(g) \otimes \beta(g) = \mu\mu'\rho(g) = \lambda(g)\rho(g)$$

for some scalar $\lambda(g)$. Then λ is a one-dimensional representation of G, and so is trivial as G is perfect. The final statement follows from Lemma 3.2. ∎

Next we show that for locally finite groups, tensor decomposability is a local property.

LEMMA 3.4: *Let G be a locally finite group, K be a field, and V be a finite-dimensional, irreducible , tensor indecomposable KG-module. Then there exists a finite subgroup F of G such that the restriction V_F of V to F is irreducible and tensor indecomposable.*

Proof: We may clearly assume that V is faithful for G, in which case G is countable [21, 9.5]. Let A be the subalgebra of $\mathrm{End}_K V = E$ spanned by the images of the elements of G. Then A can be spanned by the images of the elements of some finite subgroup G_1 of G. It follows that V is irreducible for G_1, and hence for any subgroup of G containing G_1. Further, if $E_1 = \mathrm{End}_{KG_1} V$, then

$$E_1 = \mathrm{End}_{KX} V \tag{3}$$

for any subgroup X of G containing G_1.

Let $n = \dim V$, and assume that there is no finite subgroup F of G, containing G_1, such that V_F is tensor indecomposable. Then we can write $G = \bigcup_{i=1}^{\infty} G_i$, where

$$G_1 \leq G_2 \leq \cdots \tag{4}$$

is a tower of finite subgroups of G. For each i, there exist KG_i-modules U_i, W_i, of dimension > 1, such that V is KG_i-isomorphic to $U_i \otimes W_i$. We can clearly choose integers r, s such that, for infinitely many values of i, we have $\dim U_i = r$ and $\dim W_i = s$. Deleting from (4) the terms for which these equalities do not hold, and renumbering the rest, we may assume that they hold for all i.

Let $\rho : G \to GL_n(K)$ be a matrix representation afforded by V. Then for $i = 1, 2, \ldots$, there exists $g_i \in GL_n(K)$ and representations $\sigma_i : G_i \to GL_r(K)$, $\tau_i : G_i \to GL_s(K)$ such that

$$\rho(g)^{g_i} = \sigma_i(g) \otimes \tau_i(g) \qquad (g \in G_i), \tag{5}$$

where the right-hand side is a Kronecker product of matrices. Since ρ is irreducible on G_1, so are σ_i and τ_i. As G_1 has only finitely many irreducible representations over K (up to equivalence), we may choose irreducible representations α and β of G_1 such that, for infinitely many values of i, σ_i restricts on G_1 to a representation equivalent to α and τ_i to one equivalent to β. After deleting terms of (4) and renumbering as before, we may assume that all σ_i and τ_i restrict in this way. We may also take $g_1 = 1$ in (5).

We now have

$$\rho(g) = \alpha(g) \otimes \beta(g) \qquad (g \in G_1). \tag{6}$$

Further, if $i > 1$, then there exist elements $u_i \in GL_r(K)$ and $v_i \in GL_s(K)$ such that $\sigma_i(g) = \alpha(g)^{u_i}$ and $\tau_i(g) = \beta(g)^{v_i}$ if $g \in G_1$. Putting $w_i = u_i \otimes v_i$, we have from (5) and (6) that

$$\rho(g)^{g_i w_i^{-1}} = \alpha(g) \otimes \beta(g) = \rho(g),$$

if $g \in G_1$. This tells us that $g_i w_i^{-1} \in E_1$. By (3), $g_i w_i^{-1} \in \mathrm{End}_{KG_i} V$, which allows us to replace g_i by w_i in (5). Then, conjugating by w_i^{-1} and using $\sigma_i^{u_i^{-1}}$ instead of σ_i and $\tau_i^{v_i^{-1}}$ instead of τ_i, we may assume that $g_i = 1$ and σ_i and τ_i restrict on G_1 to α and β respectively.

If $j \geq i$, the new version of (5) shows that $\sigma_j \otimes \tau_j$ and $\sigma_i \otimes \tau_i$ agree on G_i. It follows that $\sigma_j |_{G_i} = \lambda_{ij} \sigma_i$ for some one-dimensional representation λ_{ij} of G_i. Let $\mathbf{S}_i$ be the set of all representations of G_i of the form $\lambda \sigma_i$ for some linear representation λ of G_i. Then the above shows that restriction maps $\mathbf{S}_j$ into $\mathbf{S}_i$ if $j \geq i$. The sets $\mathbf{S}_i$, together with restriction maps, form an inverse system of finite non-empty sets, and as is well known, such a system has non-empty inverse limit. Thus, we can choose elements $\phi_i \in \mathbf{S}_i$ $(i = 1, 2, \ldots)$, consistent under restriction. Then there exists a representation $\phi : G \to GL_r(K)$ such that $\phi\mid_{G_i} = \phi_i$ for $i = 1, 2, \ldots$ Since $\phi_i \in \mathbf{S}_i$, there exists a representation $\psi_i : G_i \to GL_s(K)$ such that $\rho(g) = \phi_i(g) \otimes \psi_i(g)$ if $g \in G_i$. The argument just used shows that the ψ_i are consistent under restriction, and so they are common restrictions of a representation $\psi : G \to GL_s(K)$. Then clearly $\rho = \sigma \otimes \psi$. This contradicts the assumed tensor indecomposability of ρ and proves the lemma. ∎

The next result is the main one of this section. Recall that the infinitesimally irreducible representations of $\overline{G}$ are those polynomial representations which induce irreducible representations of the Lie algebra. Equivalently, they are those whose highest weight λ satisfies $0 \leq <\lambda, \alpha_i> \leq p-1$, where the α_i form a system of fundamental roots of $\overline{G}$ [4].

THEOREM 3.5: *Let $\overline{G}$ be simply connected, and F be a Frobenius map in A. Let $G = \overline{G}(F, \mathbf{n})$, and let $\rho : G \to GL_n(\overline{K})$ be an irreducible representation. Then there exist infinitesimally irreducible representations $\sigma_i : G \to GL_{n_i}(\overline{K})$ and field automorphisms $\phi_i \in \Phi$ $(1 \leq i \leq r)$ such that ρ is equivalent to the restriction to G of $\sigma_1\phi_1 \otimes \ldots \otimes \sigma_r\phi_r$.*

Proof: Let $G_i = \overline{G}^{F^{n_i}}$, so that $G = \bigcup_{i=1}^{\infty} G_i$. Now we may write ρ as a tensor product of tensor indecomposable representations, and thus assume that ρ itself is tensor indecomposable (and absolutely irreducible). By Lemma 3.2, $\rho_i = \rho_{G_i}$ is absolutely irreducible and tensor indecomposable for sufficiently large i, and we may assume that this holds for all i. By Steinberg's Theorem [17, Theorem 43, p.217] ρ_i can be extended to an irreducible polynomial representation $\overline{\rho}_i$ of $\overline{G}$. Clearly $\overline{\rho}_i$ is tensor indecomposable, and so by Steinberg's tensor product theorem, $\overline{\rho}_i$ is equivalent to a representation $\theta_i F_0^{m_i}$, where θ_i is an infinitesimally irreducible representation of $\overline{G}$ and F_0 is the Frobenius map corresponding to the p-th power map on $\overline{K}$. The set of infinitesimally irreducible representations of $\overline{G}$ is finite, so some such representation θ occurs infinitely often among the θ_i, and by passing to a subsequence and renumbering, we may assume that $\theta_i = \theta$ for all i. In a similar way we may assume that the maps $F_0^{m_i}$ all agree on G_1, and replacing ρ by a conjugate under $GL_n(\overline{K})$, that ρ and $\theta F_0^{m_1}$ agree on G_1.

Let $i > 1$. Then there exists $g_i \in GL_n(\overline{K})$ such that

$$g_i^{-1}\overline{\rho}_i(g)g_i = \theta F_0^{m_i}(g)$$

for all $g \in \overline{G}$. Now restricting to G_1 via G_i, we see that $\overline{\rho}_i(g) = \rho(g)$ for all $g \in G_1$. By Schur's Lemma, g_i is a scalar matrix, so in fact

$$\rho(g) = \overline{\rho}_i(g) = \theta F_0^{m_i}(g) \tag{7}$$

for all $g \in G_i$.

We may assume that θ is not the trivial representation of $\overline{G}$, in which case its kernel is contained in the centre Z of $\overline{G}$. From (7) we obtain

$$F_0^{m_i+1}(g) \equiv F_0^{m_i}(g) \qquad (\operatorname{mod} Z) \tag{8}$$

if $g \in G_i$. Since G_i is perfect, we can replace this congruence by equality. Now let K_i be the field over which G_i is defined, that is, $K(\overline{G}^{F^{n_i}})$ in the notation of the last section. By applying the improved version of (8) as g runs over a suitable root subgroup of G_i, we see that as automorphisms of $\overline{K}$, $F_0^{m_i+1}$ and $F_0^{m_i}$ have the same restriction to K_i. Therefore there exists a field automorphism $\phi \in \Phi$ that induces $F_0^{m_i}$ on G_i, and we finally obtain from (7) that $\rho = \theta\phi$. This completes the proof.

COROLLARY 3.6: *With the hypotheses of the previous theorem, every irreducible $\overline{K}$-representation of $\overline{G}(F, \mathbf{n})$ extends to $\overline{G}$.*

4. Proof of Theorem A: the case $G = \overline{G}$.

Before beginning this proof, we need some straightforward and probably well known facts about periodic linear groups. If G is such a group, we write $F(G)$ for the product of its normal nilpotent subgroups, and $E(G)$ for the group generated by its quasisimple subnormal subgroups. In fact $F(G)$ is nilpotent, by a result of Gruenberg [21, 8.2]. It is well known and easy to see that any two quasisimple subnormal subgroups of a group commute elementwise, and so $E(G)$ is the product of a finite number of quasisimple normal subgroups of itself, and any two of them commmute elementwise. We put $F^*(G) = F(G)E(G)$, the generalized Fitting subgroup of G, as in the finite case.

LEMMA 4.1: *Let G be a periodic linear group. Then*
(i) *Every non-trivial quotient of G contains either a non-trivial abelian characteristic subgroup or a non-abelian simple subnormal subgroup.*
(ii) $F^*(G) \geq C_G(F^*(G))$.

Proof: (i) Let H be such a quotient. By a result of Kargapolov [21, 9.30], there is a bound k to the number of non-abelian factors in any (subnormal) series of H. Now each finite soluble section of H is covered by a finite soluble subgroup of G. For such a section is isomorphic to a quotient A/B, where A and B are subgroups of G. We choose a finite subgroup X of G of minimal order such that $A = BX$. Then $X \cap B$ is in the Frattini subgroup of X and so is nilpotent, whence we see that X is soluble, as required. From this, a theorem of Zassenhaus [21, 3.7] gives us an upper bound d to the derived lengths of soluble sections of H. Suppose now that H has no non-trivial characteristic abelian subgroup. Since H is locally finite, the join of its soluble subnormal subgroups is locally soluble, and therefore soluble, since we have a bound on the derived lengths of soluble sections of H. Therefore H has no non-trivial soluble subnormal subgroups. Now put $H = H_1$, and having obtained

$$H = H_1 \triangleright H_2 \triangleright \cdots \triangleright H_n \neq 1,$$

put $K = H_n^{(d)}$. We have $K \neq 1$, and note that $K = K'$. If K is simple we stop. Otherwise we take any non-trivial proper normal subgroup of K as H_{n+1}. Then K/H_{n+1} is non-abelian, as $K = K'$. Since we have a bound on the number of non-abelian factors in a series of H, this process must lead after a finite number of steps to a non-abelian simple subnormal subgroup of H.

(ii) This now follows as in the finite case. Let $C = C_G(F^*(G))$, and suppose that $C \not\leq F = F^*(G)$. Then $C/C \cap F$ is non-trivial, and by (i), it contains a non-trivial subgroup $D/C \cap F$ that is either abelian and characteristic, or non-abelian simple subnormal. In the first case, D is nilpotent and normal in G, and in the second, $D = D'(C \cap F)$ and D' is quasisimple. In either case we find that $D \leq F$, a contradiction. ∎

LEMMA 4.2: *Let $\mathbf{n} \in \mathrm{N}$. If $\overline{G}$ has adjoint type, then $N_{\overline{G}}(\overline{G}(F,\mathbf{n})') = \overline{G}(F,\mathbf{n})$ and $\overline{G}(F,\mathbf{n})/\overline{G}(F,\mathbf{n})'$ is a finite abelian group isomorphic to a subgroup of the fundamental group of $\overline{G}$. In general, if $\overline{G}_{ad}$ is the adjoint group corresponding to $\overline{G}$, then the natural map of $\overline{G}$ onto $\overline{G}_{ad}$ maps $N_{\overline{G}}(\overline{G}(F,\mathbf{n})')/\overline{G}(F,\mathbf{n})'$ isomorphically onto $\overline{G}_{ad}(F,\mathbf{n})/\overline{G}_{ad}(F,\mathbf{n})'$.*

Proof: Let $G = \overline{G}(F,\mathbf{n})'$, $N = N_{\overline{G}}(G)$, and let G_{ad}, N_{ad} be the corresponding groups formed from $\overline{G}_{ad}$. The natural map $\pi : \overline{G} \to \overline{G}_{ad}$ maps G onto G_{ad}

[18, 12.6] and has central kernel Z. Therefore $\pi^{-1}(G_{ad}) = GZ$, and $\pi^{-1}(N_{ad}) = N_{\overline{G}}(GZ) = N$. Therefore π maps N onto N_{ad} and G onto G_{ad}, giving an isomorphism $N/G \cong N_{ad}/G_{ad}$.

Therefore, we now assume that $\overline{G}$ has adjoint type. Let $\overline{B}$ be the standard Borel subgroup of $\overline{G}$, $\overline{T}$ the standard maximal torus, and $\overline{U}$ the unipotent radical of $\overline{B}$. Also let $B = \overline{B} \cap G$, $T = \overline{T} \cap G$, and $U = \overline{U} \cap G$. Then U is a Sylow p-subgroup of G, $B = N_G(U)$, and $B = UT$. Since the Sylow p-subgroups of G are conjugate [21, 9.10], the Frattini argument gives $N = GN_N(U)$. Write $M = N_N(U)$. Then $M \cap G = B \triangleleft M$. Now T is a Hall p'-subgroup of B and the Hall p'-subgroups of B are conjugate [21, 9.22], and so a second Frattini argument gives $M = BL$, where $L = N_M(T)$. Hence $N = GL$.

Now U and $\overline{U}$ have the same nilpotency class, while any two distinct conjugates of $\overline{U}$ intersect in a group of smaller class [10]. It follows that $N_{\overline{G}}(U) \leq N_{\overline{G}}(\overline{U})$, and so $L \leq \overline{B}$. Hence $L \leq N_{\overline{B}}(T) = \overline{T}C_{\overline{U}}(T)$. It is well known that $C_{\overline{U}}(T) = 1$ (and follows for example from [10, 2.2]), and so $N = GN_{\overline{T}}(G)$. The elements of $\overline{T}$ have the form $h(\chi)$, where χ is any $\overline{K}$-valued character of the root lattice P. Let $K = K(\mathbf{n})$. Suppose that G is untwisted. Checking the conjugation action on root subgroups, we see that the condition for $h(\chi)$ to normalize G is that the values of χ lie in K. Thus in this case, $N = G\overline{T}_K = \overline{G}_K$. The last equality follows from the Bruhat decomposition. The group T consists of all K-valued characters of P that extend to K-valued characters of the full weight lattice Q. Thus, $\overline{T}_K/T$ is isomorphic to a group of K-valued characters of the fundamental group Q/P. In the other cases, $G = C_{\overline{G}_K}(\sigma)$ for some twisting automorphism σ. When there are roots of different lengths, we can use [17, Lemma 64, p.183] to see that $N = G$. In the other cases, χ must be self-conjugate in the sense of [6, p.238] and much the same argument as in the untwisted case applies. ∎

Proof of Theorem A: the case $G = \overline{G}$. Let $\overline{G}_{sc}$ be the universal cover of $\overline{G}$, and $\pi : \overline{G}_{sc} \to \overline{G}$ be the canonical projection. Then $\pi^{-1}(H)$ is dense in $\overline{G}_{sc}$, since π maps closed subgroups to closed subgroups. Combining this with Lemma 4.2, we see that it suffices to prove the result in the simply connected case. Let X^c denote the Zariski closure of a subset X of G, let $F = F(H)$, and $E = E(H)$. We refer to [21, Chapter 5] for basic facts about the Zariski topology. If J is any normal nilpotent subgroup of H, then J^c is nilpotent and normal in $H^c = \overline{G}$. Hence J^c is contained in the centre Z of $\overline{G}$, and so in particular, $F \leq Z$. Clearly $H \not\leq Z$, so by Lemma 4.1, $E \neq 1$. Let $E = E_1 \dots E_r$, where the E_i are pairwise commuting quasisimple subnormal subgroups of H, and $r \geq 1$. Let $M = N_H(E_1)$. Since the E_i are the unique quasisimple subnormal subgroups of H, they are permuted by conjugation in H, so $|H : M| \leq r$. Let $H = Ms_1 \cup \cdots \cup Ms_t$, where the s_j are elements of H and $t \leq r$. Then $\overline{G} = H^c = M^c s_1 \cup \cdots \cup M^c s_t$, so M^c is a closed subgroup of finite index in $\overline{G}$. Since $\overline{G}$ is connected, $M^c = \overline{G}$. Since $E_1^c \triangleleft M^c$, It follows that $E_1^c = \overline{G}$. But E_1 centralizes $E_2 \dots E_r$, and since centralizers are closed, so does $E_1^c = \overline{G}$. Therefore $r = 1$, which tells us that $E_1 \triangleleft H$. We claim now that it suffices to deal with the case $H = E_1$. For having done so, we shall know in the general case that E_1 is conjugate under $\operatorname{Aut}\overline{G}$ to some $\overline{G}(F, \mathbf{n})'$, and so H will be conjugate to a subgroup of $N_{\overline{G}}(\overline{G}(F, \mathbf{n})')$ containing $\overline{G}(F, \mathbf{n})'$.

Thus, we now assume that H is quasisimple. By the classification of simple periodic linear groups, we may identify $H/H \cap Z$ with a group of Lie type. That is, there exist a simple algebraic group $\overline{H}_{ad}$ of adjoint type, a Frobenius map D on $\overline{H}_{ad}$, an element $\mathbf{n} \in \mathbf{N}$, and an isomorphism

$$\alpha : \overline{H}_{ad}(D, \mathbf{n})' \to H/H \cap Z. \tag{9}$$

Let $\overline{H}_{sc}$ be the universal cover of $\overline{H}_{ad}$, with canonical projection $\pi : \overline{H}_{sc} \to \overline{H}_{ad}$. Then D lifts to a Frobenius map E on $\overline{H}_{sc}$, and $\pi : \overline{H}_{sc}(E, \mathbf{n}) \to \overline{H}_{ad}(D, \mathbf{n})'$ is a universal central extension (see [17], or use a local argument based on the finite case if twisting is present). Hence $\alpha\pi$ lifts to an epimorphism

$$\beta : \overline{H}_{sc}(E, \mathbf{n}) \to H \tag{10}$$

with central kernel.

Now let $\gamma : \overline{G} \to GL_m(\overline{K})$ be a non-trivial irreducible polynomial representation of minimal degree. Since H is dense in $\overline{G}$, γ is irreducible on H. We now claim that, for the same reason,

$$\gamma_H \textit{ is tensor indecomposable.} \tag{11}$$

For if not, then there exist $m_1, m_2 > 1$ such that $m_1 m_2 = m$ and $\gamma(H)$ is conjugate in $GL_m(\overline{K})$ to a subgroup of $GL_{m_1}(\overline{K}) \otimes GL_{m_2}(\overline{K})$. Without loss of generality, $\gamma(H) \leq GL_{m_1}(\overline{K}) \otimes GL_{m_2}(\overline{K}) = T$. Since T is closed in $GL_m(\overline{K})$ and H is dense in $\overline{G}$, it follows that $\gamma(\overline{G}) \leq T$. But then, by Lemma 3.3, γ is equivalent to the tensor product of polynomial representations of degrees m_1 and m_2, contradicting the minimality of its degree. This gives us (11). One could argue alternatively at this point that any irreducible representation of $\overline{G}$ is equivalent to a tensor product of polynomial representations twisted by field automorphisms, according to [3], and so obtain a contradiction to the minimality of the degree of γ; however the present argument is more elementary.

Now $\gamma\beta : \overline{H}_{sc}(E,\mathbf{n}) \to GL_m(\overline{K})$ is an irreducible representation, and by (11), it is tensor indecomposable. By Theorem 3.5, there exists an irreducible polynomial (in fact, infinitesimally irreducible) representation σ of $\overline{H}_{sc}$ and a field automorphism ψ of $\overline{H}_{sc}$ such that $\gamma\beta$ and $\sigma\psi$ are equivalent representations of $\overline{H}_{sc}(E,\mathbf{n})$. We replace γ by a suitable conjugate to obtain

$$(12) \qquad \gamma\beta(x) = \sigma\psi(x) \qquad \text{if } x \in \overline{H}_{sc}(E,\mathbf{n}).$$

The image of σ contains $\gamma(H)$, which is dense in $\gamma(\overline{G})$, and since $\sigma(\overline{H}_{sc})$ is closed in $GL_m(\overline{K})$, we have $\sigma(\overline{H}_{sc}) \geq \gamma(\overline{G})$. But also $\overline{H}_{sc}(E,\mathbf{n})$is dense in $\overline{H}_{sc}$, and since ψ preserves $\overline{H}_{sc}(E,\mathbf{n})$, σ maps it to $\gamma(H) \leq \gamma(\overline{G})$. It follows that $\sigma(\overline{H}_{sc}) = \gamma(\overline{G})$.

We now see that σ induces an isomorphism of abstract groups $\overline{G}_{ad} \cong \overline{H}_{ad}$. From [17, Theorem 31,p.167] we see that $\overline{G}_{ad}$ and $\overline{H}_{ad}$ are isomorphic as algebraic groups, except if $\overline{K}$ has characteristic 2 and one of $\overline{G}_{ad}$ has type B_l, the other C_l. Now $\overline{H}_{ad}$ was chosen simply as a simple algebraic group of adjoint type containing a copy of $H/H \cap Z$ of the form $\overline{H}_{ad}(D,\mathbf{n})$, so except perhaps in the excluded case, we may now take $\overline{H} = \overline{G}$. But even in that case, we can do so, since $B_l(K) \cong C_l(K)$ for all fields of characteristic 2, so $\overline{G}_{ad}$ contains a suitable copy of H if H has this type, and similarly if H happens to be a Suzuki group.

We now have surjective homomorphisms $\sigma, \gamma : \overline{G} \to \gamma(\overline{G})$ such that $\sigma(\overline{G}(E,\mathbf{n})) = \gamma(H)$. From this point we are only interested in group homomorphisms and automorphisms, and need not worry about their polynomial nature. Let $\nu : \gamma(\overline{G}) \to \overline{G}_{ad}$ be a surjective homomorphism, and let $\lambda = \nu\sigma$, $\mu = \nu\gamma$. Then λ, $\mu : \overline{G} \to \overline{G}_{ad}$ are surjective homomorphisms with the same kernel Y, the centre of $\overline{G}$. and $\lambda(\overline{G}(E,\mathbf{n})) = \mu(H)$. We can think of $\lambda : \overline{G} \to \overline{G}_{ad}$ as a universal central extension and so obtain an automorphism ϕ of $\overline{G}$ such that $\mu\phi = \lambda$. Then we have $\phi(\overline{G}(E,\mathbf{n}))Y = HY$, and taking derived groups gives $\phi(\overline{G}(E,\mathbf{n})) = H$. This completes the proof in the algebraically closed case. ∎

5. Completion of proof of Theorem A

We begin with an extension of a result well known for finite groups.

LEMMA 5.1: *Let G be a locally finite group, K be a field, and $\rho, \sigma : G \to GL_n(K)$ be representations. Suppose that L is an extension field of K, and that ρ and σ are equivalent over L. Then they are equivalent over K.*

Proof: For each finite $F \le G$, let E_F be the centralizer of $\rho(F)$ in $M_n(K)$. Then E_F is a K-algebra, and if F_1 is a finite subgroup of G containing F, then $E_F \ge E_{F_1}$. Thus, dimension considerations tell us that we may choose a finite subgroup F of G such that $E_F = E_{F_1}$ whenever $F_1 \ge F$ and F_1 is finite. By the finite case of the result we are trying to prove, [7, p.200] there exists $g \in GL_n(K)$ such that

$$g^{-1}\rho(x)g = \sigma(x) \tag{13}$$

whenever $x \in F$.

Let D be a finite subgroup of G containing F. By the same token, there exists $h \in GL_n(K)$ such that

$$h^{-1}\rho(x)h = \sigma(x)$$

for all $x \in D$. Restricting this to F and comparing with (13), we find that $hg^{-1} \in E_F = E_D$. It follows that (13) holds for all $x \in D$, and hence for all $x \in G$, as required. ∎

We also require the following facts about irreducible $\overline{K}$-representations of $\overline{G}$. They are well known, though it seems difficult to find adequate documentation. A polynomial representation σ of $\overline{G}$ is **defined over** $\mathbb{F}_p$, if the entries of the matrices $\sigma(g)$ ($g \in \overline{G}$) are polynomials with coefficients in $\mathbb{F}_p$ in the entries of g.

LEMMA 5.2: *Let $\sigma : \overline{G} \to GL_n(\overline{K})$ be an irreducible polynomial representation. Then*

(i) *σ is equivalent to a representation defined over $\mathbb{F}_p$.*

(ii) *If K is any subfield of $\overline{K}$, and σ is injective and defined over $\mathbb{F}_p$, then $\sigma(\overline{G}_K) = \sigma(\overline{G})_K$, where the suffix K denotes the group of K-rational points.*

(iii) *If $\overline{G}$ is simple and σ is defined over $\mathbb{F}_p$, then $\sigma((\overline{G}_K)') = (\sigma(\overline{G})_K)'$.*

Proof: (i) See [20, p.679].

(ii) If θ is any automorphism of $\overline{K}$, and we think of θ as acting componentwise on matrices, then clearly $\theta\sigma(g) = \sigma\theta(g)$ for any $g \in \overline{G}$. Since σ is injective, it follows that g is θ-invariant if and only if $\sigma(g)$ is. Applying this as θ varies over the Galois group of $\overline{K}$ over K, we find that g is K-rational if and only if $\sigma(g)$ is.

(iii) We may clearly take σ to be non-trivial. Let Z be its kernel, let $H = \overline{G}/Z$, and let π be the canonical map $\overline{G} \to H$. Then $\sigma = \tau\pi$, where $\tau : H \to GL_n(\overline{K})$ is an irreducible polynomial representation, also defined over $\mathbb{F}_p$. Now τ is injective, and so, using (ii) and [18, 12.6], we find

$$\sigma((\overline{G}_K)') = \tau\pi((\overline{G}_K)') = \tau((H_K)') = (\tau(H)_K)' = (\sigma(\overline{G})_K)'.$$

Conclusion of proof of Theorem A: Let $G = \overline{G}(E, \mathbf{k})'$, where E is a Frobenius map in A and $\mathbf{k} \in \mathbf{N}$. We have a dense subgroup H of G, and have to show that it is conjugate in Aut G to one of standard type. As in the case $G = \overline{G}$, we may assume that $\overline{G}$ is simply connected and H is quasisimple. Since G is dense in $\overline{G}$, so is H, and since the result is known when $G = \overline{G}$, we find that H is conjugate in Aut $\overline{G}$ to some $\overline{G}(F, \mathbf{m}) = G_1$. By Lemma 2.2, $G = \overline{G}(F, \mathbf{n})$, for some $\mathbf{n} \in \mathbf{N}$ such that $\mathbf{m} \mid \mathbf{n}$. Let $K = K(\mathbf{n})$ and $K_1 = K(\mathbf{m})$. By Lemma 2.4,

$$K_1 \leq K. \tag{14}$$

Hence

$$H \leq G \leq \overline{G}_K \quad \text{and} \quad G_1 \leq \overline{G}_K. \tag{15}$$

Note that $\overline{G}_K = \overline{G}(K)$ here as $\overline{G}$ is simply connected. Now we repeat the argument used when $G = \overline{G}$. Let $\beta : G_1 \to H$ be an isomorphism of groups, which we already know exists, and let $\gamma : \overline{G} \to GL_m(\overline{K})$ be a non-trivial irreducible polynomial representation of minimal degree. By Lemma 5.2, we may take γ to be defined over $\mathbb{F}_p$, and we do so. As before, we have the irreducible representation $\gamma\beta$ of G_1; it is tensor indecomposable and so there exist an infinitesimally irreducible representation σ of $\overline{G}$ and a field automorphism ψ of $\overline{G}$ such that $\gamma\beta$ and $\sigma\psi$ are equivalent over $\overline{K}$, as representations of G_1. Since γ and σ are both defined over $\mathbb{F}_p$, (15) tells us that both of these are K-representations. By Lemma 5.1, there exists $g \in GL_m(K)$ such that

$$g^{-1}\gamma\beta(x)g = \sigma\psi(x)$$

for all $x \in G_1$. Let δ be the representation $x \mapsto g^{-1}\gamma(x)g$ of $\overline{G}$. Then

$$\delta\beta(x) = \sigma\psi(x) \tag{16}$$

for all $x \in G_1$. As when $G = \overline{G}$, , using the density of H and G_1 in $\overline{G}$, we deduce that $\delta(\overline{G}) = \sigma(\overline{G})$. Using Lemma 5.2(iii) and the fact that g is K-rational, we have

$$\delta(\overline{G}_K) = g^{-1}\gamma(\overline{G}_K)g = ((g^{-1}\gamma(\overline{G})g)_K)' = (\delta(\overline{G})_K)' = (\sigma(\overline{G})_K)' = \sigma(\overline{G}_K),$$

That is, δ and σ both map $\overline{G}_K$ onto the same group M. Also, $\delta(H) = \sigma(G_1)$. Let L be the adjoint Chevalley group over K of the same type as $\overline{G}_K$, and let $\nu : M \to L$ be an epimorphism. Let $\lambda = \nu\sigma$ and $\mu = \nu\delta$. Then $\lambda(G_1) = \mu(H)$, and as in the algebraically closed case, we find that there is an automorphism of $\overline{G}_K$ mapping G_1 to H. Thus, the proof is complete if $G = \overline{G}_K$, that is, if G is untwisted.

Suppose now that G is twisted, so that all the F^{n_i} induce the same non-trivial symmetry of the Dynkin diagram. Since $\mathbf{m} \mid \mathbf{n}$, we find that for infinitely many i, F^{n_i} is a power of some F^{m_j}, which means that G_1 is twisted in the same way as G.

We are now in virtually the same situation as that considered in [10, §3]. Namely, G is a quasisimple twisted group of Lie type and H is a subgroup of it of the same type. The differences are that the fields involved are infinite and the groups need not be of adjoint type. However, precisely the same arguments apply and complete the proof. ∎

Proof of Corollary A1: By Theorem A, we may assume that

$$\overline{G}(F,\mathbf{m})' \leq H \leq \overline{G}(F,\mathbf{m}) \cap G,$$

where $\mathbf{m} \in \mathbf{N}$ and $\mathbf{m} \mid \mathbf{n}$. Note that now, by Lemma 4.2, $H_1 = \overline{G}(F,\mathbf{m}) \cap G$. Let σ be the element of A given by Lemma 2.3, such that $C_{\overline{G}}(\sigma) = \overline{G}(F,\mathbf{m})$ and $\Phi\sigma = \Phi F$. Since $< \Phi, \sigma >$ is an abelian group, σ commutes with F and so leaves G invariant. In case (i) we take τ to be the restriction of σ to G and use Lemma 2.5. In case (ii), construct $\mathbf{r} \in \mathbf{N}$ as in Lemma 2.6, with $r_1 = 1$. Put $H_i = \overline{G}(F, r_i\mathbf{m}) \cap G$, and let τ_i be the restriction of σ to H_i. We have $H_i' = \overline{G}(F, r_i\mathbf{m})'$ and so $H_i' < H_{i+1}'$ by Lemma 2.2. Certainly, therefore, $H_i < H_{i+1}$. By Lemma 2.5, τ_i has finite order. ∎

Proof of Corollary A3: We deduce this from Corollary A1 by applying the following lemma. In case (i) of that Corollary, we take $D = G$. In case (ii), we choose i such that $< H, X > \leq H_i$ and take $D = H_i$. ∎

LEMMA 5.3: *Let $G \leq GL_n(K)$, where K is any field, and suppose that ϕ is an automorphism of finite order of G. Then G can be embedded in some $GL_m(K)$ in such a way that in this new embedding, $C_G(\phi)$ is Zariski-closed in G.*

Proof: Let G^* be the semidirect product $G < \phi >$ Since $|G^* : G| < \infty$, we can realize G^* as a subgroup of some $GL_m(K)$, simply by inducing up the given representaton of G. Since centralizers in linear groups are Zariski closed, the result follows. ∎

6. Confined subgroups of simple linear groups

We first collect together some basic facts about confined subgroups, and then use them to prove Theorem B.

LEMMA 6.1: *Let H be a subgroup of a group G.*
(i) *If $H \leq K \leq G$ and H is confined in G, then K is confined in G.*
(ii) *Let Ω be the left G-set G/H. Then H is confined in G if and only if there is a finite subset X of $G \setminus 1$ such that every point of Ω is fixed by some member of X.*

Proof: These are both easy to prove. We thank D. Evans for pointing out (ii) to us. ∎

LEMM 6.2: *Let H be a subgroup of the infinite locally finite group G. The following three conditions are equivalent.*
(i) *H is not confined in G.*
(ii) *Every finite subgroup of G has a regular orbit on $\Omega = G/H$.*
(iii) *Every finite subgroup of G has an infinite number of regular orbits on $\Omega = G/H$.*

Proof: If $F \leq G$, then the F-orbit gH is regular if and only if $F \cap gHg^{-1} = 1$, so (i) and (ii) are equivalent. Trivially, (iii) implies (ii). To see the converse, suppose that F is a finite subgroup of G having a finite number n of regular orbits on Ω. Since G is infinite, we may choose a finite subgroup E of G containing F, such that $|E : F| > n$. By (ii), E has a regular orbit on Ω, and this breaks up into $|E : F|$ regular F-orbits, a contradiction. ∎

The following is a slightly stronger form of a statement made near the end of the Introduction.

LEMMA 6.3: *Let H be a subgroup of the infinite locally finite group G. Of the following statements, each implies the next.*
(i) *H is not confined in G.*
(ii) *Each finite subgroup of G has infinitely many regular orbits on $\Omega = G/H$.*
(iii) *If R is any commutative ring, U is any non-zero R-free RH-module, and F is any finite subgroup of G, then $U^G = RG \otimes_{RH} U$ contains a free RF-direct summand of infinite rank.*
(iv) *If U is any non-zero RH-module, then U^G is a faithful RG-module.*

Proof: Lemma 5.3 tells us that (i) implies (ii). We can quote Mackey's Theorem to see that (ii) implies (iii), or argue directly as follows. We split up Ω into F-orbits. If gH belongs to a regular F-orbit, then the elements fg ($f \in F$) form a set of coset representatives for the cosets in that orbit, and $\oplus_{f\in F} fg \otimes U$ is a non-zero free RF-module. The direct sum of these over all regular F-orbits is an RF-direct summand of Ω. Since any non-zero ideal of RG has non-zero intersection with RF, for some finite $F \leq G$, it is clear that (iii) implies (iv).

The next lemma is crucial in the proof of Theorem B. If H is a subgroup of a group G, we write

$$H_G = \cap_{g\in G} H^g$$

for the *core* of H in G.

LEMMA 6.4: *Let G be a connected linear group, and H be a closed subgroup of G. Then H is confined in G if and only if $H_G \neq 1$.*

Proof: Of course, if $H_G \neq 1$, then H is confined in G, whether or not G is linear. For the converse, we will obtain a contradiction from the supposition that H is confined in G but $H_G = 1$. Since H is confined, there exists a finite subset F of $G \setminus 1$ such that $H^g \cap F \neq \emptyset$ for all $g \in G$. For each $x \in F$, the map $\phi_x : g \mapsto gxg^{-1}$ is a continuous map of G into itself, and so $\phi_x^{-1}(H)$ is closed. Since $x \notin H_G$, it follows that $\phi_x^{-1}(H) \neq G$. Since G is connected, we deduce that $\cup_{x\in F}\phi_x^{-1}(H) \neq G$. Choosing $g \in G \setminus \cup_{x\in F}\phi_x^{-1}(H)$, we have $gxg^{-1} \notin H$ for all $x \in F$, whence $F \cap H^g = \emptyset$. This is the contradiction sought. ∎

Now we are ready to prove Theorem B.

Proof of Theorem B: We have an infinite simple periodic linear group G and a proper subgroup H of it, and have to prove that H is not confined in G. Suppose that it is . Let C be the closure of H in G. Then by Lemma 6.1, C is confined in G. By Lemma 6.4, C has non-trivial core, and since G is simple, $C = G$. In other words, H is dense in G.

By the classification of infinite simple periodic linear groups referred to in the Introduction , we can identify G with a group $\overline{G}(F, \mathbf{n})'$, where $\overline{G}$ is a simple algebraic group of adjoint type over $\overline{K}$, F is a Frobenius map on $\overline{G}$, and $\mathbf{n} \in \mathbf{N}$. By Theorem A, $M = H'$ is simple and if $H_1 = N_G(M)$, then H_1/H is finite abelian. Since H is confined, there exists a finite subset X of $G \setminus 1$ such that

$$H^g \cap X \neq \emptyset \tag{17}$$

for all $g \in G$. We have $H_1 \neq G$, and so we may assume that $X \not\subseteq H_1$. Since G is simple, there exist elements $g_1, \ldots g_n \in G$ such that

$$\langle H, X\rangle \leq \langle M^{g_1}, \ldots, M^{g_n}\rangle = L.$$

By Corollary A3, there exists a subgroup D of G such that $\langle H, g_1, \ldots, g_n\rangle \leq D$, and D can be viewed as a linear group with H_1 as a closed subgroup. We have $H \leq H_1 \cap L < L$, and $\langle H, X\rangle \leq L$. Since L is generated by infinite simple groups, it is connected under any representation as a linear group. Also, thinking of L as a linear group under the new embedding of D, $H_1 \cap L$ is a proper closed subgroup of L. Finally, the core J of $H_1 \cap L$ in L is trivial. For otherwise, it either contains M or centralizes it. In the first case we find that X normalizes M, contrary to assumption, and in the second, since M is dense in G, we find that J is central in G and so is trivial. Thus, Lemma 6.4 tells us that $H_1 \cap L$ is not confined in L. By Lemma 6.1 (i), H is also not confined in L. Therefore, as $X \subseteq L$, there exists $g \in L$ such that $H^g \cap X = \emptyset$, contrary to (17) above. This establishes Theorem B.

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